9.5: Matrices and Matrix Operations

- **Definitions and Dimensions of Matrices:**
  - A rectangular array of numbers, functions or parameters is called a matrix.
  - A matrix with **dimension** $m \times n$ has m rows and n columns. When we refer to ij-entry, we mean the entry on the ith row and jth column.
  - We denote matrices with capital letters. The following is the **general representation** of an $m \times n$ matrix A:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- **Matrix addition and subtraction:**

  Let A and B be $m \times n$ matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

  Then $A \pm B = [a_{ij} \pm b_{ij}] = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{21} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$

  Note that matrix addition/subtraction has to be done on matrices with **same dimensions** and it produces a matrix of the same dimensions with entries that are addition/subtraction of the same corresponding entries of A and B.

- **Scalar multiplication:**

  Let A be a $n \times m$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and k be a number. Then the **Scalar** multiplication of the matrix A by k is

$$kA = k[a_{ij}] = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]$$

- **Some properties of matrix addition/subtraction:**

  (1) Commutative Property Of Addition: $A + B = B + A$

  (2) Associative Property of Addition $A + (B + C) = (A + B) + C$

- **The identity of matrix addition** The $m \times n$ matrix $O_{m\times n}$ is the following matrix:

$$O_{m\times n} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

  Note that $O_{m\times n} + A_{m\times n} = A_{m\times n}$.

- **Matrix Multiplication:** The matrix multiplication is only defined if the number of columns of the first matrix is equal to number of rows of the second matrix.

  Let A of be an $m \times n$ matrix and B be an $n \times k$ matrix, then the dimension of matrix AB is $m \times k$.

  ($A_{m\times n}B_{n\times k} = AB_{m\times k}$.) The ith row, jth entry of AB is the product of ith row of A and jth column of B as described below:

  Example: Here is the multiplication of $A_{2\times 3}$ by $B_{3\times 3}$: $\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{A} \overbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}^{B}$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

  Note that the matrix BA is not defined.

- **The identity of matrix multiplication**

  The identity matrix I_n or $I_{n\times n}$ is a square matrix of size n where all entries on the major diagonal are one and all the other entries are zero.

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- **Important properties of the identity matrix**

  Given any $m \times n$ matrix A the following is true. $\quad I_mA = A \quad\quad AI_n = A$

- **Equal matrices:** Let A and B be two $m \times n$ matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

  Then $A = B$ means that $a_{ij} = b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

1. What is a general representation for a 2×3 matrix?

2. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix}$.

   (a) What is the dimension of A?

   (b) What is a_{23} entry of matrix A?

3. Determine which of the following operations is defined?

   (a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 5 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 11 & 7 \\ 5 & 2 & 1 \end{bmatrix}$

4. Perform the following matrix operations.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 11 & 1 \\ 5 & 2 & 1 \end{bmatrix}$

(b) $3\begin{bmatrix} 3 & 11 & 7 \\ 5 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 11 & 1 \\ 5 & 2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

5. Find AB and BA for the following choices of matrix A and matrix B, or say that they are not defined. In case that they are both defined, determine if $AB = BA$ or not.

   (a) $A = \begin{bmatrix} 1 & 3 \\ 5 & 0.6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 0.5 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find any of the following multiplication if it is defined.

   (a) $I_3 A$

   (b) AI_3

   (c) $I_2 A$

   (d) AI_2

7. (a) Calculate $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 6 & 1 \\ 0.5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) Express the matrix equality $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 6 & 1 \\ 0.5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in terms of three equations.

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1. Example 1: https://mediahub.ku.edu/media/MATH+-+Matrices.m4v/1_pn42m74z
2. Example 2: https://mediahub.ku.edu/media/MATH+-+Matrix+Multiplication+1.m4v/1_b0z677nu
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